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DISPLACEMENT BOUNDS FOR BEAM-COLUMNS WITH INITIAL CURVATURE SUBJECTED TO TRANSIENT LOADS†

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Abstract—Elastic beam-columns with initial curvature and pinned or clamped ends are considered. Upper bounds on the lateral displacement response to transient axial and distributed lateral loads are derived by means of energy-type functionals and some inequalities. The results are significant because of their generality.

INTRODUCTION

THE response of elastic beam-columns to transient loading is considered in this paper. The beam-columns are allowed to have some initial curvature and a combination of pinned and clamped end conditions. Time-varying axial and distributed lateral loads of a general nature are considered. In most such cases the equation of motion cannot be solved analytically and an explicit solution for the lateral displacement is not available.

An upper bound for the displacement is derived here with the use of some inequalities and an energy functional. This bound is valid for general initial shapes and applied loads. A second bound applicable to pulse axial loading is also obtained. Some examples are presented and a comparison of the upper bound with the exact displacement is made for a particular case.

The importance of these displacement bounds lies in their generality and simplicity. Some similar problems, for example, have been investigated in Refs. [1–7] by a variety of methods. The problem considered here is more general than each of these in at least one of the following four aspects: the end conditions, the initial shape, the axial load and the lateral load. For cases in which the beam-column response cannot be easily determined, these bounds furnish some quick and useful information.

EQUATION OF MOTION

Consider a linearly elastic beam-column of length L with constant values of Young's modulus E, cross-sectional moment of inertia I and mass per unit length m. Let T denote the time and X the coordinate axis along the line connecting the ends of the beam-column. The equilibrium configuration in the absence of loads is given by $W_0(X)$, while $W(X, T)$ represents the lateral displacement from $W_0(X)$. An axial load $P(T)$ and a distributed lateral load $Q(X, T)$ per unit length act on the column for $0 \leq T \leq T_f$ (see Fig. 1).

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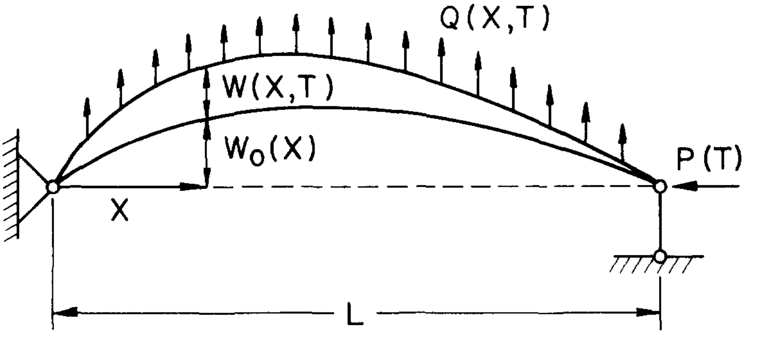

FIG. 1. Geometry of beam-column.

The displacement $W(X, T)$ is assumed to be governed by the linear equation of motion

$$m\frac{\partial^2 W}{\partial T^2}+EI\frac{\partial^4 W}{\partial X^4}+P(T)\frac{\partial^2 W}{\partial X^2}=Q(X,T)-P(T)\frac{\mathrm{d}^2 W_0}{\mathrm{d}X^2}, \qquad 0\le X\le L, T\ge 0, \tag{1}$$

where $P(T)\equiv 0$ and $Q(X,T)\equiv 0$ for $T>T_f$. In terms of the nondimensional quantities

$$\begin{aligned} p &= PL^2/EI, q = QL^3/EI, x = X/L \\ w &= W/L, w_0 = W_0/L, t = T(EI/mL^4)^{1/2}, \\ t_f &= T_f(EI/mL^4)^{1/2}, \end{aligned} \tag{2}$$

equation (1) becomes

$$\frac{\partial^4 w}{\partial x^4}+p(t)\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial t^2}=r(x,t), \qquad 0\le x\le 1, \qquad t\ge 0, \tag{3}$$

where

$$r(x,t)=q(x,t)-p(t)\frac{\mathrm{d}^2 w_0}{\mathrm{d}x^2} \tag{4}$$

and $w=w(x,t)$. The beam-column is assumed to be initially at rest, so that

$$w(x,0)=0, \qquad \frac{\partial w}{\partial t}(x,0)=0. \tag{5}$$

Three sets of end conditions are considered:

(i) pinned–pinned

$$w=\frac{\partial^2 w}{\partial x^2}=0 \quad \text{at } x=0,1;$$

(ii) clamped–clamped

$$w=\frac{\partial w}{\partial x}=0 \quad \text{at } x=0,1;$$

(iii) clamped–pinned

$$w = \frac{\partial w}{\partial x} = 0 \quad \text{at } x = 0, \qquad w = \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at } x = 1.$$

The nondimensional Euler buckling load p_e has the values (i) π^2, (ii) $4\pi^2$, (iii) $2{\cdot}046\pi^2$ for these end conditions.

In general, equation (3) is not separable and an analytic solution for $w(x, t)$ is not available. With the use of energy-type functionals, however, upper bounds on the displacement can be obtained.

DERIVATION OF DISPLACEMENT BOUNDS

(a) Consider the positive-definite functional V which is twice the sum of the kinetic energy and strain energy:

$$V = \int_0^1 \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] \mathrm{d}x. \tag{6}$$

The time rate of change of V is given by

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 2\int_0^1 \left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial t\,\partial x^2} + \frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial t^2}\right] \mathrm{d}x. \tag{7}$$

With the use of the equation of motion (3), the end conditions, and integration by parts, one can show that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -2p(t)\int_0^1 \frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial t}\mathrm{d}x + 2\int_0^1 r(x, t)\frac{\partial w}{\partial t}\mathrm{d}x \tag{8}$$

during motion of the beam-column. By Schwartz's inequality

$$\int_0^1 r(x, t)\frac{\partial w}{\partial t}\mathrm{d}x \le v(t)\left\{\int_0^1 \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x\right\}^{\frac{1}{2}} \le v(t)V^{\frac{1}{2}} \tag{9}$$

where

$$v(t) = \left\{\int_0^1 r^2(x, t)\,\mathrm{d}x\right\}^{\frac{1}{2}}. \tag{10}$$

Also,

$$2\int_0^1 \frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial t}\mathrm{d}x \le \int_0^1 \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\mathrm{d}x = V. \tag{11}$$

From (8), (9) and (11) it then follows that

$$\frac{\mathrm{d}V}{\mathrm{d}t} \le |p(t)|V + 2v(t)V^{\frac{1}{2}}. \tag{12}$$

Let $V = U^2$ with $U \ge 0$, so that (12) becomes

$$\frac{\mathrm{d}U}{\mathrm{d}t} \le \tfrac{1}{2}|p(t)|U + v(t) \tag{13}$$

and, upon integration and application of the initial conditions (5),

$$U(t) \leq \int_0^t v(\tau) \exp\left\{\tfrac{1}{2}\int_\tau^t |p(\xi)|\,d\xi\right\} d\tau. \tag{14}$$

This inequality provides a bound on U. In order to obtain a bound on $|w(x, t)|$, consider the inequality [8]

$$\int_0^1 \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \geq k^2 w^2 \tag{15}$$

where k has the values (i) 6·928, (ii) 13·86, (iii) 10·09 for the three sets of end conditions. Combination of (14) and (15) then leads to the bound

$$|w(x, t)| \leq (1/k) \int_0^t v(\tau) \exp\left\{\tfrac{1}{2}\int_\tau^t |p(\xi)|\,d\xi\right\} d\tau. \tag{16}$$

If the maximum possible value of $|w(x, t)|$ for $0 \leq x \leq 1$ and $t \geq 0$ is denoted by w_{max}, it follows that

$$w_{max} \leq (1/k) \int_0^{t_f} v(t) \exp\left\{\tfrac{1}{2}\int_t^{t_f} |p(\xi)|\,d\xi\right\} dt \tag{17}$$

where $v(t)$ is defined by (10).

Inequality (17) furnishes an upper bound on the displacement of the beam-column in terms of the transient loads $p(t)$ and $q(x, t)$ and the initial curvature d^2w_0/dx^2.

(b) A second useful displacement bound may be obtained with the use of the functional

$$S = \int_0^1 \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 - p(t)\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] dx \tag{18}$$

if $p(t)$ is a continuous pulse loading with the following properties:

$$\begin{aligned} &0 \leq p(t) < p_e && \text{for} \quad 0 \leq t \leq t_f, \\ &p(t) \equiv 0 && \text{for} \quad t > t_f, \\ &\dot{p}(t) \geq 0 && \text{for} \quad 0 \leq t \leq t_d, \\ &\dot{p}(t) \leq 0 && \text{for} \quad t_d < t \leq t_f, \end{aligned} \tag{19}$$

where $\dot{p} \equiv dp/dt$. In this case S is positive-definite, since [8]

$$\int_0^1 \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \geq p_e \int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 dx, \tag{20}$$

and

$$\frac{dS}{dt} = -\dot{p}(t)\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 dx + 2\int_0^1 r(x, t)\frac{\partial w}{\partial t}\, dx. \tag{21}$$

From (9), (19) and (20) it follows that the second integral on the right side of (21) satisfies the inequality

$$\int_0^1 r(x, t)\frac{\partial w}{\partial t}\, dx \leq v(t) S^{1/2} \tag{22}$$

and the first integral is non-positive for $0 \le t \le t_d$. Therefore,

$$\frac{\mathrm{d}S}{\mathrm{d}t} \le 2v(t)S^{1/2} \quad \text{for} \quad 0 \le t \le t_d \tag{23}$$

and integration leads to

$$[S(t)]^{\frac{1}{2}} \le \int_0^t v(\tau)\,\mathrm{d}\tau \quad \text{for } 0 \le t \le t_d \tag{24}$$

with the right side maximum for $t = t_d$.

For $t_d \le t \le t_f$ one can show that

$$-\dot{p}(t)\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x \le 2\gamma(t)S \tag{25}$$

where

$$\gamma(t) = \frac{-\dot{p}(t)}{2[p_e - p(t)]} \ge 0, \tag{26}$$

and hence

$$\frac{\mathrm{d}S}{\mathrm{d}t} \le 2\gamma(t)S + 2v(t)S^{\frac{1}{2}} \quad \text{for} \quad t_d \le t \le t_f\,. \tag{27}$$

Inequality (27) is similar to (12), and integration here yields

$$[S(t)]^{\frac{1}{2}} \le [S(t_d)]^{\frac{1}{2}} \exp\left\{\int_{t_d}^t \gamma(\tau)\,\mathrm{d}\tau\right\} + \int_{t_d}^t v(\tau)\exp\left\{\int_\tau^t \gamma(\xi)\,\mathrm{d}\xi\right\}\mathrm{d}\tau \quad \text{for} \quad t_d \le t \le t_f\,. \tag{28}$$

This expression can be simplified, since

$$\int_\tau^t \gamma(\xi)\,\mathrm{d}\xi = \ln\left\{\frac{p_e - p(t)}{p_e - p(\tau)}\right\}^{\frac{1}{2}}, \tag{29}$$

to give

$$[S(t)]^{\frac{1}{2}} \le [p_e - p(t)]^{\frac{1}{2}}\left\{\frac{1}{[p_e - p(t_d)]^{\frac{1}{2}}}\int_0^{t_d} v(\tau)\,\mathrm{d}\tau + \int_{t_d}^t \frac{v(\tau)}{[p_e - p(\tau)]^{\frac{1}{2}}}\,\mathrm{d}\tau\right\} \quad \text{for} \quad t_d \le t \le t_f\,, \tag{30}$$

where (24) with $t = t_d$ has been used. A weaker but simpler bound is given by

$$[S(t)]^{\frac{1}{2}} \le \left\{\frac{p_e - p(t)}{p_e - p(t_d)}\right\}^{\frac{1}{2}} \int_0^t v(\tau)\,\mathrm{d}\tau \quad \text{for} \quad t_d \le t \le t_f \tag{31}$$

which follows from the fact that $p(t)$ is maximum at $t = t_d$.

For $t > t_f$, $\mathrm{d}S/\mathrm{d}t = 0$ and $S(t) = S(t_f)$. Upper bounds on $[S(t)]^{\frac{1}{2}}$ for all t are, therefore, given by the right sides of (30) and (31) when $t = t_f$. Displacement bounds then follow from

$$S \ge \frac{p_e - p(t_d)}{p_e}\int_0^1 \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x \ge \frac{[p_e - p(t_d)]}{p_e}k^2 w^2. \tag{32}$$

For example, combination of (31) at $t = t_f$ with (32) leads to the upper bound

$$w_{\max} \leq (1/k)\{p_e/[p_e - p(t_d)]\} \int_0^{t_f} v(t)\,\mathrm{d}t \tag{33}$$

on the displacement magnitude $|w(x, t)|$, where $v(t)$ is defined by (10) and $p(t)$ is a pulse load which satisfies conditions (19).

EXAMPLES

(a) In some practical applications, exact expressions for the loads and initial curvature may not be known. Suppose that only the following information is given:

$$|p(t)| \leq p_m, \qquad |q(x,t)| \leq q_m, \qquad \left|\frac{\mathrm{d}^2 w_0}{\mathrm{d}x^2}\right| \leq a_m \quad \text{for} \quad 0 \leq t \leq t_f$$
$$p(t) \equiv 0, \quad q(x,t) \equiv 0 \quad \text{for} \quad t > t_f\,. \tag{34}$$

Then for $0 \leq t \leq t_f$

$$v(t) \leq \max_{0 \leq x \leq 1} |r(x,t)| \leq q_m + p_m a_m \tag{35}$$

and inequality (17) provides the displacement bound

$$w_{\max} \leq (2/k)[a_m + (q_m/p_m)](\mathrm{e}^{p_m t_f/2} - 1) \tag{36}$$

in terms of the extreme values of $p(t)$, $q(x, t)$ and $\mathrm{d}^2 w_0/\mathrm{d}x^2$.

(b) Consider the case $q(x, t) \equiv 0$, so that only the axial load $p(t)$ is acting on the beam-column. In this case

$$v(t) = \alpha |p(t)| \tag{37}$$

where

$$\alpha = \left\{\int_0^1 \left(\frac{\mathrm{d}^2 w_0}{\mathrm{d}x^2}\right)^2 \mathrm{d}x\right\}^{\frac{1}{2}} \tag{38}$$

and (17) may be integrated to give

$$w_{\max} \leq (2\alpha/k)\left[\exp\left\{\tfrac{1}{2}\int_0^{t_f} |p(t)|\,\mathrm{d}t\right\} - 1\right]. \tag{39}$$

For a pulse load of the form (19), inequality (33) is also applicable and yields the displacement bound

$$w_{\max} \leq (\alpha/k)\{p_e/[p_e - p(t_d)]\} \int_0^{t_f} |p(t)|\,\mathrm{d}t. \tag{40}$$

(c) If $p(t) \equiv 0$, then inequality (17) becomes

$$w_{\max} \leq (1/k) \int_0^{t_f} \left\{\int_0^1 q^2(x,t)\,\mathrm{d}x\right\}^{\frac{1}{2}} \mathrm{d}t. \tag{41}$$

In the case of a separable lateral load

$$q(x,t) = \phi(t)f(x) \tag{42}$$

the bound (41) may be written as

$$w_{\max} \leq (1/k)\left\{\int_0^1 f^2(x)\,dx\right\}^{\frac{1}{2}} \int_0^{t_f} |\phi(t)|\,dt. \tag{43}$$

With no axial load present, $w(x,t)$ is independent of the initial shape $w_0(x)$; however, the total displacement of the beam-column from the x axis is given by $w(x,t)+w_0(x)$.

Consider, for example, a pinned-pinned beam subjected to the lateral load

$$q(x,t) = \begin{cases} q_0 \sin \pi x \sin \omega_0 t & \text{for} \quad 0 \leq t < \pi/\omega_0 \\ 0 & \text{for} \quad t \geq \pi/\omega_0 \end{cases} \tag{44}$$

where ω_0 is the fundamental vibration frequency of the beam. For this particular case the exact displacement can be calculated, and one can show that the maximum displacement over all time has the value 0·16 q_0/ω_0 as compared to the upper bound 0·20 q_0/ω_0 given by (43).

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Абстракт—Рассматриваются упругие балки-колонны с начальным искрывлением и защемленными концами. Определяются верхние пределы поведения горизонтальных перемещений, вследствие нестационарных осевых сил или распредённых горизонтальных нагрузок, путём функционалов энергического типа и некоторых неравенств. Результаты оказываются важными вследствие их обобщения.